§14.7

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$

$$\nabla f = (2x + y + 3,\ x + 2y - 3).$$

$$\nabla f = (0,0) \iff \begin{cases} 2x + y + 3 = 0 \\ x + 2y - 3 = 0 \end{cases}$$

$$\iff (x, y) = (-3, 3).$$

$\therefore$ $(-3, 3)$ is the only critical point.

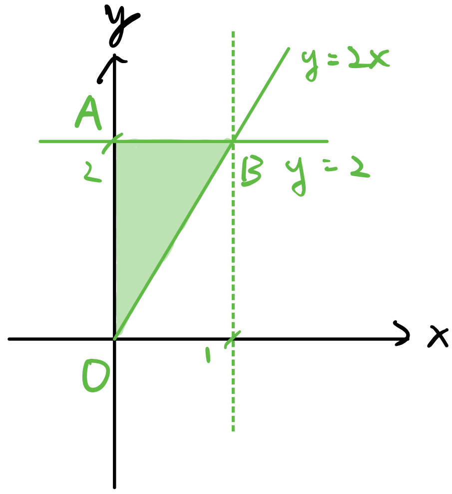
$f_{xx} = 2 > 0$, $f_{yy} = 2$, $f_{xy} = 1$

$f_{xx} f_{yy} - f_{xy}^2 = 3 > 0$

$\therefore$ $(-3, 3)$ is local min.

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant



The absolute extrema appears in boundary of domain or local extrema.

(i) On the segment OA,

$$\left(\begin{array}{c} x=0 \\ 0 \le y \le 2 \end{array} \right)$$

$$\begin{aligned} f(0,y) &= y^2 - 4y + 1 \\ &= (y^2 - 4y + 4) - 3 \\ &= (y-2)^2 - 3 \end{aligned}$$

$\therefore$ on OA,

f attains min when $y = 2$, $f(0,2) = -3$

f attains max when $y = 0$, $f(0,0) = 1$

(ii) On AB $(0 \le x \le 1,\ y = 2)$,

$$\begin{aligned} f(x,2) &= 2x^2 - 4x - 3 \\ &= 2(x^2 - 2x + 1) - 5 \\ &= 2(x-1)^2 - 5 \end{aligned}$$

$\therefore$ f attains min when $x = 1$, $f(1,2) = -5$

f attains max when $x = 0$, $f(0,2) = -3$

(iii) on OB $(0 \le x \le 1,\ y = 2x)$,

$$f(x, 2x) = 2x^2 - 4x + 4x^2 - 8x + 1$$
$$= 6(x^2 - 2x + 1) - 5$$
$$= 6(x-1)^2 - 5$$

$\therefore$ f attains min when $x = 1$, $f(1,2) = -5$

f attains max when $x = 0$, $f(0,0) = 1$

(iv) Critical point in the interior:

$$\nabla f = (4x - 4,\ 2y - 4)$$
$$\nabla f = (0,0) \iff (x,y) = (1,2).$$

Note $(1,2)$ is on the boundary, not in interior.

$\therefore$ No local extreme inside the interior.

$\therefore$ Abs. max. : $(0,0)$ with $f(0,0) = 1$

Abs. min : $(1,2)$ with $f(1,2) = -5$.

43. Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that

a. $f_x = 2x - 4y$ and $f_y = 2y - 4x$

b. $f_x = 2x - 2$ and $f_y = 2y - 4$

c. $f_x = 9x^2 - 9$ and $f_y = 2y + 4$

Describe your reasoning in each case.

(a). $\nabla f = (0,0) \iff (x,y) = (0,0)$.

$\therefore (0,0)$ is the only critical point.

$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -4$

$f_{xx} f_{yy} - f_{xy}^2 = -12 < 0$

$\therefore (0,0)$ is a saddle point.

(b). $\nabla f = (0,0) \iff (x,y) = (1,2)$.

$\therefore (1,2)$ is the only critical point.

$f_{xx} = 2, \; f_{yy} = 2, \; f_{xy} = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 4 > 0$

$\therefore (1,2)$ is a local min.

(c). $\nabla f = (0,0) \iff (x,y) = (1,-2)$

or $(x,y) = (-1,-2)$

$f_{xx} = 18x \quad f_{yy} = 2 \quad f_{xy} = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 36x$.

$\therefore (1,-2)$ is a local min

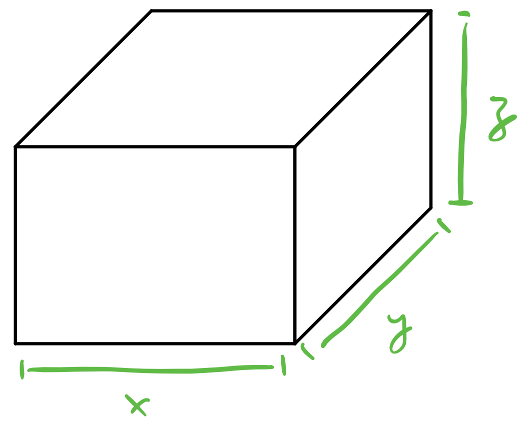
$(-1,-2)$ is a saddle point.

58. Among all closed rectangular boxes of volume 27 cm^3, what is the smallest surface area?

$x, y, z > 0$

volume $= xyz = 27$

$\Rightarrow z = \dfrac{27}{xy}$



surface area

$= 2xy + 2xz + 2yz$

$= 2xy + 2x \cdot \dfrac{27}{xy} + 2y \cdot \dfrac{27}{xy}$

$= 2xy + \dfrac{54}{y} + \dfrac{54}{x}$

Denote the surface area by $f(x, y)$.

$f_x = 2y - \dfrac{54}{x^2}$ $\qquad$ $f_y = 2x - \dfrac{54}{y^2}$

$f_x = 0 \Rightarrow y = \dfrac{27}{x^2}$

$f_y = 0 \Rightarrow 2x - \dfrac{54}{27^2} x^4 = 0$

$\Rightarrow x\left(1 - \dfrac{1}{27} x^3\right) = 0$

$\Rightarrow x = 0$ (reject)

or $x = 3$

$\therefore y = 3$

$\therefore (3, 3)$ is only critical point.

$$f_{xx} = \frac{108}{x^3}, \quad f_{yy} = \frac{108}{y^3}, \quad f_{xy} = 2$$

$$f_{xx}(3,3) = 4, \quad f_{yy}(3,3) = 4$$

$$\therefore \ f_{xx} f_{yy} - f_{xy}^2 = 12 > 0$$

$\therefore$ $(3,3)$ is a local min.
with surface area $= 54$.

Since there is no boundary point,
$x = y = z = 3$ gives the smallest surface area ($= 54\,cm^2$).

Find the absolute maximum and minimum values of the following functions on the given curves.

63. Function: $f(x, y) = xy$

Curves:

i) The line $x = 2t, \quad y = t + 1$

ii) The line segment $x = 2t, \quad y = t + 1, \quad -1 \le t \le 0$

iii) The line segment $x = 2t, \quad y = t + 1, \quad 0 \le t \le 1$

(i).
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$
$$= y\cdot 2 + x\cdot 1$$
$$= 4t+2.$$
$$\frac{df}{dt} = 0 \iff t = -0.5.$$
$$\left.\frac{d^2f}{dt^2}\right|_{t=-0.5} = 4$$

Note the line has no boundary points.

$\therefore$ $t=-0.5$ gives $f(-1, 0.5) = -0.5$, which is Abs. min.

No Abs. max. as $(x,y)\to(\infty,\infty)$ or $(x,y)\to(-\infty,-\infty)$ gives $f\to\infty$

(ii). For the two ends of the line segment,

$t=-1 \Rightarrow (x,y) = (-2, 0)$

$f(-2,0) = 0$

$t=0 \Rightarrow (x,y) = (0, 1)$

$f(0,1) = 0$

$\therefore$ Abs. max : 0, when $t=0$ or $t=1$

Abs. min. : -0.5, when $t = -0.5$.

(iii). $t=0$: above

$t=1 \Rightarrow (x,y) = (2,2)$

$f(2,2) = 4$

$\therefore$ Abs. max. : 4, $t=1$

Abs. min. : 0, $t=0$.

In Exercises 69–74, you will explore functions to identify their local extrema. Use a CAS to perform the following steps:

a. Plot the function over the given rectangle.

b. Plot some level curves in the rectangle.

c. Calculate the function's first partial derivatives and use the CAS equation solver to find the critical points. How do the critical points relate to the level curves plotted in part (b)? Which critical points, if any, appear to give a saddle point? Give reasons for your answer.

d. Calculate the function's second partial derivatives and find the discriminant $f_{xx}f_{yy} - f_{xy}^{\ 2}$.

e. Using the max-min tests, classify the critical points found in part (c). Are your findings consistent with your discussion in part (c)?

70. $f(x, y) = x^3 - 3xy^2 + y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

Recommandation :

Desmos / GeoGebra : Plot Graph

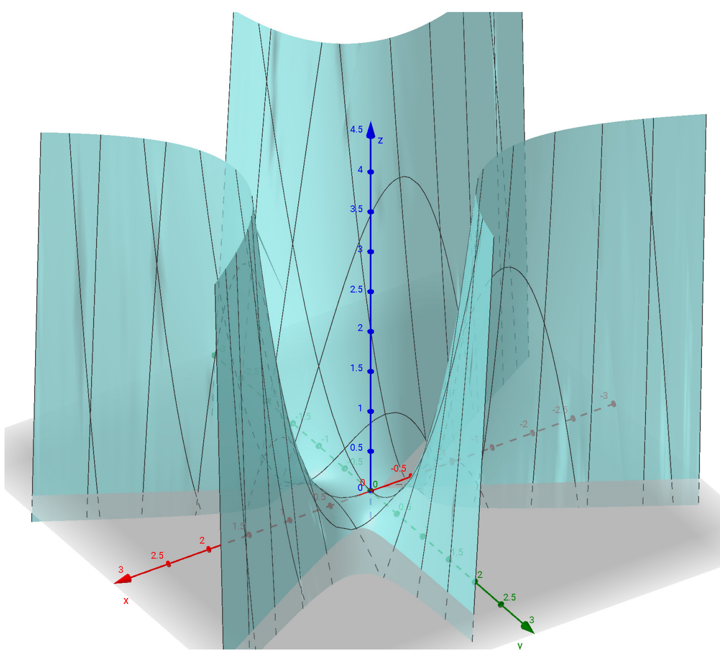
Symbolab / Wolfram : Calculation

Matlab : Plot Graph, calculation

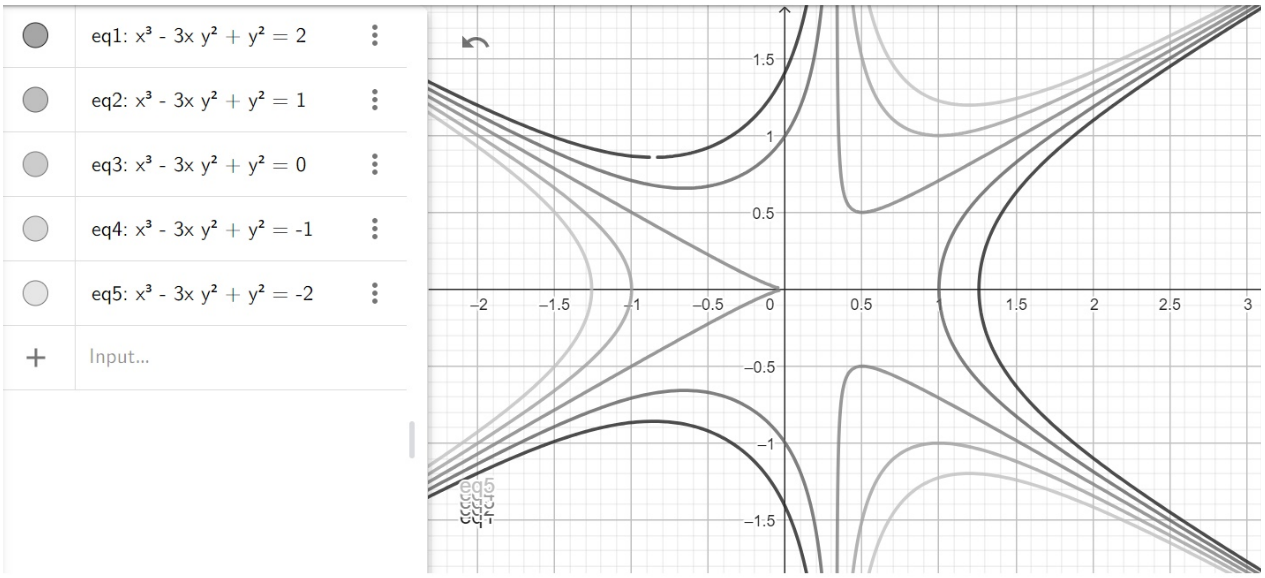
(Matlab will be taught in MATH2221)

(a) Plot :

(GeoGebra 3D)



(b): level Curve: (GeoGebra)



(c):

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial f}{\partial y} = -6xy + 2y = 2y(-3x+1)$$

$$f_x = 0 \Rightarrow x^2 = y^2$$
$$\Rightarrow x = \pm y$$

$x = y$, $f_y = 0$ $\Rightarrow 0 = 2y(-3y+1)$

$$\Rightarrow y = 0 \text{ or } y = 1/3$$

$x = -y$, $f_y = 0$ $\Rightarrow 0 = 2y(3y+1)$

$$\Rightarrow y = 0 \text{ or } y = -1/3$$

$\therefore$ critical points: $(0,0)$, $(1/3, 1/3)$, $(1/3, -1/3)$

(d). $f_{xx} = 6x$

$f_{xy} = -6y$

$f_{yy} = -6x+2$

$$f_{xx}f_{yy} - f_{xy}^2 = -36x^2 + 12x - 36y^2$$

$$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(0,0)} = 0$$

$$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(1/3,\,1/3)} = -4 < 0$$

$$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(1/3,\,-1/3)} = -4 < 0$$

(e): $(0,0)$: inconclusive by min-max test

$(1/3, 1/3)$, $(1/3, -1/3)$: saddle points.

§14.8

1. Extrema on an ellipse Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

function : $f(x,y) = xy$

constraint : $g(x,y) = x^2 + 2y^2 - 1 = 0$

$\nabla f = (y, x) \quad \nabla g = (2x, 4y)$

$\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{cases} y = 2\lambda x \\ x = 4\lambda y \end{cases}$$

$\Rightarrow \quad x = 8\lambda^2 x, \quad y = 8\lambda^2 y$

$\Rightarrow \quad x = y = 0$ (rej. as $0^2 + 2 \cdot 0^2 \neq 1$)

or $8\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{2\sqrt{2}}$

$\therefore \quad x = \pm\sqrt{2}\, y$

$x^2 + 2y^2 = 1 \Rightarrow 2y^2 + 2y^2 = 1$

$\Rightarrow \quad y = \pm \frac{1}{2}$.

$\therefore$ The extreme points are :

max : $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ or $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$, value $= \frac{1}{2\sqrt{2}}$

min : $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ or $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, value $= \frac{-1}{2\sqrt{2}}$

6. Constrained minimum Find the points on the curve $x^2y = 2$ nearest the origin.

distance between (x, y) and $(0, 0)$

is $\sqrt{x^2 + y^2}$

Note minimizing $\sqrt{x^2 + y^2}$

$\iff$ minimizing $x^2 + y^2$

define $f(x, y) = x^2 + y^2$.

$g(x, y) = x^2 y - 2$.

$\nabla f = (2x, 2y)$

$\nabla g = (2xy, x^2)$

$\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{cases} 2x = 2\lambda x y \\ 2y = \lambda x^2 \end{cases}$$

$\Rightarrow$ $x = y = 0$ (rej. as $0^2 \cdot 0 \neq 2$)

or $x = \pm \frac{\sqrt{2}}{\lambda}$, $y = \frac{1}{\lambda}$

$x^2 y = 2 \Rightarrow \frac{2}{\lambda^3} = 2 \Rightarrow \lambda = 1$

$\therefore$ $(+\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$

are points with shortest distance (both $= \sqrt{3}$)

§14.9

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

4. $f(x, y) = \sin x \cos y$

$f(0,0) = 0$

$f_x(x,y) = \cos x \cos y \Rightarrow f_x(0,0) = 1$

$f_y(x,y) = -\sin x \sin y \Rightarrow f_y(0,0) = 0$

$f_{xx}(x,y) = -\sin x \cos y \Rightarrow f_{xx}(0,0) = 0$

$f_{xy}(x,y) = -\cos x \sin y \Rightarrow f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -\sin x \cos y \Rightarrow f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = -\cos x \cos y \Rightarrow f_{xxx}(0,0) = -1$

$f_{xxy}(x,y) = \sin x \sin y \Rightarrow f_{xxy}(0,0) = 0$

$f_{xyy}(x,y) = -\cos x \cos y \Rightarrow f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = \sin x \sin y \Rightarrow f_{yyy}(0,0) = 0$

$\therefore$ quadratic approx.:

$0 + 1(x) + 0(y) + \frac{0}{2!}(x)^2 + 2\cdot\frac{0}{2!}(xy) + \frac{0}{2!}y^2$

$= x$

Cubic approx.:

$x + \frac{1}{3!}(-1)x^3 + \frac{3}{3!}(0)x^2y + \frac{3}{3!}(-1)xy^2 + \frac{1}{3!}(0)y^3$

$= x - \frac{1}{6}x^3 - \frac{1}{2}xy^2$

6. $f(x, y) = \ln(2x + y + 1)$

$$f(0,0) = 0$$

$$f_x = \frac{2}{2x+y+1} \Rightarrow f_x(0,0) = 2$$

$$f_y = \frac{1}{2x+y+1} \Rightarrow f_y(0,0) = 1$$

$$f_{xx} = \frac{-4}{(2x+y+1)^2} \Rightarrow f_{xx}(0,0) = -4$$

$$f_{xy} = \frac{-2}{(2x+y+1)^2} \Rightarrow f_{xy}(0,0) = -2$$

$$f_{yy} = \frac{-1}{(2x+y+1)^2} \Rightarrow f_{yy}(0,0) = -1$$

$$f_{xxx} = \frac{16}{(2x+y+1)^3} \Rightarrow f_{xxx}(0,0) = 16$$

$$f_{xxy} = \frac{8}{(2x+y+1)^3} \Rightarrow f_{xxy}(0,0) = 8$$

$$f_{xyy} = \frac{4}{(2x+y+1)^2} \Rightarrow f_{xyy}(0,0) = 4$$

$$f_{yyy} = \frac{2}{(2x+y+1)^3} \Rightarrow f_{yyy}(0,0) = 2.$$

$\therefore$ quadratic approx.:

$$0 + 2(x) + 1(y) + \frac{1}{2}(-4x^2 + 2(-2)xy + (-1)y^2)$$

$$= 2x + y - \frac{1}{2}(4x^2 + 4xy + y^2)$$

$$= 2x + y - \frac{1}{2}(2x + y)^2$$

Cubic approx.:

$$2x+y-\frac{1}{2}(2x+y)^2$$

$$+\frac{1}{3!}\left(16x^3+3\cdot 8x^2y+3\cdot 4xy^2+2\cdot y^3\right)$$

$$=2x+y-\frac{1}{2}(2x+y)^2$$

$$+\frac{1}{3}\left((2x)^3+3(2x)^2y+3(2x)y^2+y^3\right)$$

$$=2x+y-\frac{1}{2}(2x+y)^2+\frac{1}{3}(2x+y)^3$$